Regularity and Normality of the Secant Variety to a Projective Curve

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Abstract

For a smooth curve of genus g embedded by a line bundle of degree at least $2g+3$ we show that the ideal sheaf of the secant variety is 5-regular. This bound is sharp with respect to both the degree of the embedding and the bound on the regularity. Further, we show that the secant variety is projectively normal for the generic embedding of degree at least $2g+3$.

AMS Subject Classification (2000): 14F17, 14H60, 14N05

Key words: Secant Variety; Regularity; Normality

1 The Result

We work over an algebraically closed field of characteristic 0. Recall the classical theorem of Castelnuovo:

Theorem 1 *Let $C \subset \mathbb{P}^n$ be a linearly normal embedding of a smooth curve of genus g by a line bundle L with $\mathrm{c}_1(L) \geq 2g+1$. Then $\mathcal{I}_C$ is 3-regular and (equivalently) $C \subset \mathbb{P}^n$ is projectively normal.* □

The following extension was proved for $a=2$ by J. Rathmann [10] and was proved in general by the author [14, 4.2] (see also [2]).

Theorem 2 *Let $C \subset \mathbb{P}^n$ be a linearly normal embedding of a smooth curve of genus g by a line bundle L with $\mathrm{c}_1(L) \geq 2g+3$. Then $\mathcal{I}_C^a$ is $(2a+1)$-regular.* □

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Considering C to be the zeroth secant variety to itself, and denoting the first secant variety by Σ_1, in this work we obtain what is perhaps a more natural extension.

Theorem 3 *Let $C \subset \mathbb{P}^n$ be a linearly normal embedding of a smooth curve of genus g by a line bundle L with $\mathrm{c}_1(L) \geq 2g+3$. Then $\mathcal{I}_{\Sigma_1}$ is 5-regular and at least for the generic such embedding $\Sigma_1 \subset \mathbb{P}^n$ is projectively normal.* $\square$

We expect that $\Sigma_1 \subset \mathbb{P}^n$ is always projectively normal under these hypotheses. The most significant difficulty in the proof of Theorem 3 is that Σ_1 has rational singularities if and only if C itself is rational. Thus the usual technique [3] of blowing up a variety and studying line bundles on the blow-up rather than ideal sheaves on $\mathbb{P}^n$ requires significant care.

Remark 4. Note that in $\mathbb{P}^4$, the secant variety to a non-degenerate elliptic curve of degree 5 is a quintic hypersurface, and that the secant variety to a non-degenerate genus 2 curve of degree 6 is an octic hypersurface; hence Theorem 3 is sharp. $\square$

Because the kth secant variety Σ_k to an elliptic normal curve $C \subset \mathbb{P}^{2+2k}$ is a hypersurface of degree $2k+3$, Theorem 3 and Remark 4 suggest:

Conjecture 5 *Let $C \subset \mathbb{P}^n$ be a linearly normal embedding of a smooth curve of genus g by a line bundle L. If $\mathrm{c}_1(L) \geq 2g+1+2k$, $k \geq 0$, then $\mathcal{I}_{\Sigma_k}$ is $(2k+3)$-regular and Σ_k is projectively normal.* $\square$

We combine this with a previous conjecture of the author [13, 3.10] to form the following Green-Lazarsfeld type conjecture. Following [5] we say that a closed projective scheme $X \subset \mathbb{P}^n$ satisfies $\mathrm{N}_{k,p}$ if the ideal of X is generated by forms of degree k and the syzygies are linear for $p-1$ steps.

Conjecture 6 *Let $C \subset \mathbb{P}^n$ be a linearly normal embedding of a smooth curve of genus g by a line bundle L. If $\mathrm{c}_1(L) \geq 2g+1+p+2k$, $p, k \geq 0$, then Σ_k satisfies $\mathrm{N}_{k+2,p}$.* $\square$

For $k=0$ this is the famous result of [7] (see also [8]). For $g=0$ this seems well-known. For $g=1$ this was proven in [4] and in [6]. For $g=2$, calculations done by J. Sidman [11] with Macaulay 2 [9] support the conjecture for $\mathrm{c}_1(L) \leq 13$.

2 The Proof

We denote the ith secant variety to an embedded projective curve $C \subset \mathbb{P}^n$ by Σ_i. Note that $\Sigma_0 = C$.

A line bundle L on a curve C is said to be *k-very ample* if $h^0(C, L(-Z)) = h^0(C, L) - k$ for all $Z \in S^k C$. We recall (the first stages of) Bertram's 'Terracini Recursiveness' construction, which provides the geometric framework for our results.

Theorem 7 *[1, Theorem 1] Let $C \subset B_0 = \mathbb{P}(H^0(C, L))$ be a smooth curve embedded by a line bundle L. Suppose L is 4-very ample and consider the birational morphism $f : B_2 \to B_0$ which is a composition of the following blow-ups:*

$f^1 : B_1 \to B_0$ is the blow up of B_0 along Σ_0

$f^2 : B_2 \to B_1$ is the blow up along the proper transform of Σ_1

Then, the proper transform of Σ_1 in B_1 is smooth and irreducible, transverse to the exceptional divisor, so in particular B_2 is smooth. Let E_i be the proper transform in B_i of each f^i-exceptional divisor.

(Terracini recursiveness) Suppose $x \in \Sigma_1 \setminus C$. Then the fiber $f^{-1}(x) \subset B_2$ is naturally isomorphic to $\mathbb{P}(H^0(C, L(-2Z)))$, where Z is the unique divisor of degree 2 whose span contains x. If $x \in C$ the fiber $f^{-1}(x) \subset E_1 \subset B_2$ is isomorphic to the blow up of $\mathbb{P}(H^0(C, L(-2x)))$ along the image of C embedded by $L(-2x)$. □

Lemma 8 *[14, 3.2] Hypotheses and notation as above:*

(1) $\Sigma_1 \subset B_0$ is normal.
(2) $f_* \mathcal{O}_{B_2} = \mathcal{O}_{B_0}$ and $R^j f_* \mathcal{O}_{B_2} = 0$ for $j \geq 1$.
(3) $f_* \mathcal{O}_{E_i} = \mathcal{O}_{\Sigma_{i-1}}$ for $i = 1, 2$. □

Our proof proceeds by the well-known technique (Cf. [3]) of obtaining vanishings on the blow-ups, and then deducing vanishing statements on B_0. In the case of a smooth variety $X \subset \mathbb{P}^n$, one has $\mathrm{H}^i(\widetilde{\mathbb{P}^n}, \mathcal{O}(kH - aE)) = \mathrm{H}^i(\mathbb{P}^n, \mathcal{I}_X^a(k))$. The significant difficulty in the case of secant varieties is the following:

Proposition 9 *Let $C \subset \mathbb{P}^n$ be a 4-very ample embedding of a smooth curve. Then Σ_1 has rational singularities if and only if C is rational.*

PROOF: Consider the blow up $f^1 : B_1 \to \mathbb{P}^n$. By [12, 3.8] we have $E_1 \cap \widetilde{\Sigma}_1 = C \times C$ with the restriction $f^1 : E_1 \cap \widetilde{\Sigma}_1 \to C$ just projection onto one factor. From the sequence

$$0 \to \mathcal{I}_{C\times C/B_1} \to \mathcal{O}_{B_1} \to \mathcal{O}_{C\times C} \to 0$$

we see $R^2 f^1_* \mathcal{I}_{C\times C/B_1} = \mathrm{H}^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C$, which in turn implies $R^2 f^1_* \mathcal{I}_{\widetilde{\Sigma}_1/B_1} = \mathrm{H}^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C$. Because $R^2 f^1_* \mathcal{I}_{\widetilde{\Sigma}_1/B_1} = R^1 f^1_* \mathcal{O}_{\widetilde{\Sigma}_1}$, this completes the proof.
□

Thus, in the notation of Theorem 7, $R^1 f_* \mathcal{O}_{E_2} \neq 0$ for a non-rational curve and so the transfer of vanishing from B_2 to B_0 requires some care. In particular, $\mathrm{H}^2(B_1, \mathcal{I}_{\widetilde{\Sigma}_1}(k)) \neq 0$ for all $k >> 0$. Our main result is:

Theorem 10 *Let $C \subset \mathbb{P}^n$, $n \geq 5$, be a smooth curve embedded by a non-special line bundle L. Assume that C is linearly and cubically normal, and that there is a point $p \in C$ such that $L(2p)$ is 6-very ample and $L(2p - 2q)$ satisfies K_2 for all $q \in C$. Then $\mathcal{I}_{\Sigma_1}$ is 5-regular.*

Recall [12] that an embedding $C \subset \mathbb{P}^n$ satisfies K_2 if $\mathcal{I}_C(2)$ is globally generated and the Koszul syzygies are generated by linear syzygies.

Corollary 11 *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle of degree at least $2g + 3$. Then Σ_1 is 5-regular.*

PROOF: It is well-known [7] that a line bundle of degree at least $2g+3$ satisfies the hypotheses of Theorem 10. We need only mention that a rational normal curve of degree 3 has $\Sigma_1 = \mathbb{P}^3$, that the secant variety to a rational normal curve of degree 4 is a cubic hypersurface, and that the secant variety to an elliptic normal curve of degree 5 is a quintic hypersurface. □

PROOF:(of Theorem 10) By Proposition 9 we know that $f_* \mathcal{O}_{B_2}(-E_2) = \mathcal{I}_{\Sigma_1}$, that $R^2 f_* \mathcal{O}_{B_2}(-E_2) = \mathrm{H}^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C$, and that $R^i f_* \mathcal{O}_{B_2}(-E_2) = 0$ for all other i.

$\underline{\mathbf{H^1(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(4)) = 0}}$: Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, $\mathcal{G}$ a coherent sheaf on B_2. From the 5-term sequence

$$0 \to \mathrm{H}^1(\mathbb{P}^n, \mathcal{F} \otimes f_* \mathcal{G}) \to \mathrm{H}^1(B_2, f^* \mathcal{F} \otimes \mathcal{G}) \to \mathrm{H}^0(\mathbb{P}^n, \mathcal{F} \otimes R^1 f_* \mathcal{G}) \to \cdots$$

associated to the Leray-Serre spectral sequence we see that it is enough to show that $\mathrm{H}^1(B_2, \mathcal{O}_{B_2}(4H - E_2)) = 0$.

Consider the sequence

$$0 \to \mathcal{O}_{B_2}(4H - E_1 - E_2) \to \mathcal{O}_{B_2}(4H - E_2) \to \mathcal{O}_{E_1}(4H - E_2) \to 0$$

Along the fibers F_c of $E_1 \to C$ we have $\mathcal{O}_{B_2}(4H - E_2) \otimes \mathcal{O}_{F_c} = \mathcal{O}_{F_c}(-E)$. Therefore $R^i f_* \mathcal{O}_{E_1}(4H - E_2) = 0$ for $i \neq 2$, hence $\mathrm{H}^i(E_1, \mathcal{O}_{E_1}(4H - E_2)) = 0$ for $i = 0, 1$, and so it suffices to show $\mathrm{H}^1(B_2, \mathcal{O}_{B_2}(4H - E_1 - E_2)) = 0$. By [14, 3.3] we know that $\mathrm{H}^1(B_2, \mathcal{O}_{B_2}(3H - E_1 - E_2)) = 0$; pulling the Euler sequence on $\mathbb{P}^n$ up to B_2 we have

$$0 \to f^*\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{B_2}(4H - E_1 - E_2) \to \bigoplus_1^{n+1} \mathcal{O}_{B_2}(3H - E_1 - E_2) \to \mathcal{O}_{B_2}(4H - E_1 - E_2) \to 0$$

and it suffices to show $\mathrm{H}^2(B_2, f^*\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{B_2}(4H - E_1 - E_2)) = 0$.

By non-specialty of L together with cubic normality of the embedding we have $\mathrm{H}^1(B_2, \mathcal{O}_{B_2}(4H - E_1)) = \mathrm{H}^2(B_2, \mathcal{O}_{B_2}(3H - E_1)) = 0$, hence $\mathrm{H}^2(B_2, f^*\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{B_2}(4H - E_1)) = 0$. Finally, we show $\mathrm{H}^1(E_2, f^*\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{E_2}(4H - E_1)) = \mathrm{H}^1(\widetilde{\Sigma}_1, f^*\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{\widetilde{\Sigma}_1}(4H - E_1)) = 0$.

From the sequence

$$0 \to \mathcal{O}_{B_2}(3H - E_1 - E_2) \to \mathcal{O}_{B_2}(3H - E_1) \to \mathcal{O}_{E_2}(3H - E_1) \to 0$$

because we know that $\mathrm{H}^1(B_2, \mathcal{O}_{E_2}(3H - E_1)) = 0$ by cubic normality and that $\mathrm{H}^2(B_2, \mathcal{O}_{B_2}(3H - E_1 - E_2)) = 0$ by [14, 3.3], we have $\mathrm{H}^1(E_2, \mathcal{O}_{E_2}(3H - E_1)) = 0$. Again working with the Euler sequence

$$0 \to f^*\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{\widetilde{\Sigma}_1}(4H - E_1) \to \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes \mathcal{O}_{\widetilde{\Sigma}_1}(3H - E_1) \to \mathcal{O}_{\widetilde{\Sigma}_1}(4H - E_1) \to 0$$

we show

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes \mathrm{H}^0(\widetilde{\Sigma}_1, \mathcal{O}_{\widetilde{\Sigma}_1}(3H - E_1)) \to \mathrm{H}^0(\widetilde{\Sigma}_1, \mathcal{O}_{\widetilde{\Sigma}_1}(4H - E_1))$$

is surjective.

We recall [1],[12] that $\mathcal{O}_{B_1}(2H - E_1)$ is globally generated and that the restriction of the induced morphism φ to $\widetilde{\Sigma}_1$ is a $\mathbb{P}^1$ bundle $\varphi : \widetilde{\Sigma}_1 \to S^2C$; furthermore, $\widetilde{\Sigma}_1 = \mathbb{P}_{S^2C}(\varphi_* \mathcal{O}_{\widetilde{\Sigma}_1}(H))$. In particular, there is a very ample line bundle $\mathcal{O}_{S^2C}(1)$ such that $\varphi^* \mathcal{O}_{S^2C}(1) = \mathcal{O}_{\widetilde{\Sigma}_1}(2H - E_1)$. It is easy to check, again with the Euler sequence, that $\varphi_* \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes \mathcal{O}_{\widetilde{\Sigma}_1} \to E = \varphi_* \mathcal{O}_{B_1}(H)$ is surjective, hence E is globally generated. Applying φ_* to the Euler sequence from the previous paragraph yields

$$\cdots \to \varphi_* \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes E \otimes \mathcal{O}_{S^2C}(1) \to S^2E \otimes \mathcal{O}_{S^2C}(1) \to 0$$

By global generation of E and of $\mathcal{O}_{S^2C}(1)$, and the vanishing of all higher direct images, we have

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes \mathrm{H}^0(\widetilde{\Sigma}_1, \mathcal{O}_{\widetilde{\Sigma}_1}(3H - E_1)) \to \mathrm{H}^0(\widetilde{\Sigma}_1, \mathcal{O}_{\widetilde{\Sigma}_1}(4H - E_1))$$

is surjective.

$\underline{\mathbf{H^2(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(3)) = 0}}$: Again by [14, 3.3] we know that $\mathrm{H}^i(B_2, 3H - E_1 - E_2) = 0$ for $i \geq 1$. Therefore we have

$$\begin{aligned}\mathrm{H}^2(\mathcal{O}_{B_2}(3H - E_2)) &= \mathrm{H}^2(\mathcal{O}_{E_1}(3H - E_2)) \\ &= \mathrm{H}^0(C, R^2 f_* \mathcal{O}_{E_1}(3H - E_2)) \\ &= \mathrm{H}^0(C, R^2 f_* \mathcal{O}_{B_2}(3H - E_2))\end{aligned}$$

It is straightforward to check that $E_2^{2,0} = E_\infty^{2,0}$, therefore because the edge map $\mathrm{H}^2(B_2, \mathcal{O}_{B_2}(3H - E_2)) \to \mathrm{H}^0(C, R^2 f_* \mathcal{O}_{B_2}(3H - E_2))$ is a quotient [16, 5.2.6], this implies $\mathrm{H}^2(\mathbb{P}^n, f_* \mathcal{O}_{B_2}(3H - E_2)) = \mathrm{H}^2(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(3)) = 0$.

$\underline{\mathbf{H^3(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(2)) = 0}}$: The fact that $\mathrm{H}^2(\widetilde{\Sigma}_1, \mathcal{O}_{\widetilde{\Sigma}_1}(2H - E_1)) = 0$ is contained in the proof of [14, 3.6]; therefore $\mathrm{H}^3(B_2, \mathcal{O}_{B_2}(2H - E_1 - E_2)) = 0$. Consider

$$0 \to \mathcal{O}_{B_2}(2H - E_1 - E_2) \to \mathcal{O}_{B_2}(2H - E_2) \to \mathcal{O}_{E_1}(2H - E_2) \to 0$$

We have $R^i f_* \mathcal{O}_{E_1}(2H - E_2) = 0$ for $i \neq 2$ and $R^2 f_* \mathcal{O}_{E_1}(2H - E_2) = \mathrm{H}^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C(2)$. Thus $\mathrm{H}^3(E_1, \mathcal{O}_{E_1}(2H - E_2)) = \mathrm{H}^1(C, R^2 f_* \mathcal{O}_{E_1}(2H - E_2)) = 0$, and so $\mathrm{H}^3(B_2, 2H - E_2) = 0$. Further, we also have

$$\mathrm{H}^2(B_2, \mathcal{O}_{B_2}(2H - E_2)) \to \mathrm{H}^2(E_1, \mathcal{O}_{E_1}(2H - E_2))$$

is surjective, but as above

$$\mathrm{H}^2(E_1, \mathcal{O}_{E_1}(2H - E_2)) = \mathrm{H}^0(C, R^2 f_* \mathcal{O}_{B_2}(2H - E_2))$$

and, therefore, $E_2^{0,2} = E_\infty^{0,2}$. This implies that $d_3 : E_3^{0,2} \to E_3^{3,0} = E_2^{3,0}$ is the zero map, and so $E_2^{3,0} = E_\infty^{3,0}$. Thus the vanishing $\mathrm{H}^3(B_2, 2H - E_2) = 0$ implies $\mathrm{H}^3(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(2)) = 0$.

$\underline{\mathbf{H^4(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(1)) = 0}}$: Finally, $\mathrm{H}^3(\widetilde{\Sigma}_1, \mathcal{O}_{B_1}(H)) = \mathrm{H}^3(S^2C, \varphi_* \mathcal{O}_{B_1}(H)) = 0$. Therefore, as it is not hard to see $E_2^{4,0} = E_\infty^{4,0}$, we have $\mathrm{H}^4(B_2, H - E_2) = \mathrm{H}^4(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(1)) = 0$. □

From the first part of the proof, we obtain the following general statement:

Proposition 12 *Let $C \subset \mathbb{P}^n$ be a 4-very ample embedding of a smooth curve. Then $\mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_C^2(k)) \geq \mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(k))$ for $k \leq 3$. In particular, if C is also linearly normal then Σ_1 is linearly normal.*

PROOF: The proof shows

$$\begin{aligned}
\mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(k)) &= \mathrm{h}^1(B_2, \mathcal{O}_{B_2}(kH - E_2)) \\
&= \mathrm{h}^1(B_2, \mathcal{O}_{B_2}(kH - 2E_1 - E_2)) \\
&\leq \mathrm{h}^1(B_2, \mathcal{O}_{B_2}(kH - 2E_1))
\end{aligned}$$

where the vanishing $\mathrm{H}^0(E_2, \mathcal{O}_{B_2}(kH - 2E_1)) = 0$ for $k \leq 3$ comes from the structure of $\widetilde{\Sigma}_1$ as a $\mathbb{P}^1$-bundle over S^2C. The fact that $\mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_C^2(1)) = 0$ for C linearly normal is [15, 1.3.2]. □

Remark 13. By the extension of Bertram's "Terracini Recursiveness" to higher dimensions [12], Proposition 12 also holds for a 4-very ample embedding of any projective variety $X \subset \mathbb{P}^n$ as long as the embedding satisfies K_2. □

It is worth pointing out that we do not always get equality in Proposition 12:

Example 14. Let $C \subset \mathbb{P}^4$ be a rational normal curve. Then Σ_1 is a cubic hypersurface, hence $\mathrm{H}^1(\mathbb{P}^4, \mathcal{I}_{\Sigma_1}(2)) = 0$. However, we can compute directly that $\mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_C^2(2)) = 3$. □

Theorem 15 *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a non-special line bundle L. Assume that C is linearly and cubically normal, and that there is a point $p \in C$ such that $L(2p)$ is 6-very ample and $L(2p - 2q)$ satisfies K_2 for all $q \in C$. Then for the general $q \in C$, the secant variety to C under the embedding by $L(2p - 2q)$ is projectively normal.*

PROOF: Under these hypotheses, $\mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_C^2(3)) = 0$ by [14, 3.10], therefore we have $\mathrm{H}^1(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(3)) = 0$ by Proposition 12. By Theorem 10 and Proposition 12, we are left to show $\mathrm{H}^1(\mathbb{P}^n, \mathcal{I}_{\Sigma_1}(2)) = 0$. This follows in general from [14, 3.5,3.9]. In particular, the hypotheses allow us to construct a sequence of blow-ups

$$f : B_3 \to B_2 \to B_1 \to \mathbb{P}\Gamma(C, L)$$

where $B_3 \to B_2$ is the blow up of the proper transform of Σ_2. It is shown there that $R^1 f_* \mathcal{O}_{E_1}(kH - 2E_1 - 2E_2 - E_3) = 0$, which implies the generic vanishing of $\mathrm{h}^1(B_2, \mathcal{O}_{B_2}(2H - 2E_2 - E_2)$. In order to get vanishing for ALL q using this technique, one would need to show $R^i f_* \mathcal{O}_{E_3}(kH - 2E_1 - 2E_2 - E_3) = 0$ for $i \geq 2$. □

Corollary 16 *Let C be a smooth curve. For the generic $L \in \operatorname{Pic}^k C$, $k \geq 2g+3$, the secant variety $\Sigma_1 \subset \mathbb{P}\Gamma(C, L)$ is projectively normal.* $\square$

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